Verteilte Algorithmen 182.073
Quiz 1 (Ch.2 + Pre, Sample Quiz), Form: A

Name: ________________________________
Prog. code, registr. no.: ___________________
Date: ________________________________
Achieved points: _______________________

Question 1. Basics:

Answer with true or false.

_____ A configuration $C = (q_0, \ldots, q_{n-1})$ is a vector of the local states of processors $p_0, \ldots, p_{n-1}$.

_____ In the synchronous model, message delays can be arbitrary.

_____ Termination is an example of a safety property.

Question 2. Consider a spanning tree $T$ of graph $G$, with known root:

Match the properties to the trees (hint: one to one).

_____ In a spanning tree

_____ In a breadth-first search tree (BFS)

_____ In a depth-first search tree (DFS)

(a) every pair of neighbors in $G$ is on a path from the root in $T$.

(b) there is exactly one path from any node in $T$ to the root.

(c) every node at distance $d$ in $G$ is at depth $d$ in $T$.

Question 3. If $f(n) = \sqrt{n}$ and $g(n) = n \log n + O(n)$ for $n \to \infty$, then

Hint: Multiple choices possible.

1. $f(n)g(n) =$
   (a) $n^{3/2} \log n + O(n^{3/2})$
   (b) $n^{1/2} \log n^{1/2} + O(n^{1/2})$
   (c) $O(n^2)$

2. $f(n)/g(n) =$
   (a) $O(n^{-1/2})$
   (b) $o(n^{-1/2})$
   (c) $\Omega(1/\log n)$
   (d) $\Theta(n^{-1/2})$

Question 4.

Fill in the missing word(s).

1. How many ways are there to assign not necessarily distinct elements drawn from a set of $n$ elements to $k$ distinguishable boxes (exactly one element per box)?

2. How many ways are there to choose a set of $k$ (distinct) elements from a set of $n$ (distinct) elements?

Question 5. Prove or disprove in a mathematical sound way:

Write readable!
1. Using an indirect proof, show that the complement $\overline{G}$ of a not connected simple undirected graph $G$ is connected. Recall that $(x, y) \in E(G) \Leftrightarrow (x, y) \notin E(\overline{G})$.

2. Develop a recursive formula for $a_n$, which is the number of sequences for length $n \geq 1$ of elements taken from $\{0, 1\}$ that have an odd number of 0. [Hint: Consider the 2 possibilities for choosing the first element of such a sequence.]

3. [Pre], [SS21,Sample]. Using induction on $n \geq 1$, show that $\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$, for every $m \geq 0$. [Hint: Recall that $\binom{0}{0} = 1$ and $\binom{0}{1} = 0$.]
Question 1. Basics:

Answer with true or false.

True   A configuration $C = (q_0, \ldots, q_{n-1})$ is a vector of the local states of processors $p_0, \ldots, p_{n-1}$.

False   In the synchronous model, message delays can be arbitrary.

False   Termination is an example of a safety property.

Question 2. Consider a spanning tree $T$ of graph $G$, with known root:

Match the properties to the trees (hint: one to one).

(b) In a spanning tree
(c) In a breadth-first search tree (BFS)
(a) In a depth-first search tree (DFS)

(a) every pair of neighbors in $G$ is on a path from the root in $T$.
(b) there is exactly one path from any node in $T$ to the root.
(c) every node at distance $d$ in $G$ is at depth $d$ in $T$.

Question 3. If $f(n) = \sqrt{n}$ and $g(n) = n \log n + O(n)$ for $n \to \infty$, then

Hint: Multiple choices possible.

1. $f(n)g(n) =$
   - (a) $n^{3/2} \log n + O(n^{3/2})$
   - (b) $n^{1/2} \log n^{1/2} + O(n^{1/2})$
   - (c) $O(n^2)$

2. $f(n)/g(n) =$
   - (a) $O(n^{-1/2})$
   - (b) $o(n^{-1/2})$
   - (c) $\Omega(1/\log n)$
   - (d) $\Theta(n^{-1/2})$

Question 4.

Fill in the missing word(s).

1. How many ways are there to assign not necessarily distinct elements drawn from a set of $n$ elements to $k$ distinguishable boxes (exactly one element per box)? $n^k$

2. How many ways are there to choose a set of $k$ (distinct) elements from a set of $n$ (distinct) elements? $\binom{n}{k}$

Question 5. Prove or disprove in a mathematical sound way:

Write readable!

1. Using an indirect proof, show that the complement $\overline{G}$ of a not connected simple undirected graph $G$ is connected. Recall that $(x, y) \in E(G) \Leftrightarrow (x, y) \notin E(\overline{G})$.

Proof: Assume $\overline{G}$ is not connected. Then, there must be two vertices $x, y \in V(\overline{G})$ with $(x, y) \notin E(\overline{G}) \Rightarrow (x, y) \in E(G)$. Hence, $x, y$ must be in the same component in $G$. Let $z$ be a vertex in some other component of $G$, hence $(x, z) \notin E(G)$ and $(y, z) \notin E(G) \Rightarrow (x, z) \in E(\overline{G})$ and $(y, z) \in E(\overline{G})$. Those two edges form a path between $x$ and $y$ in $\overline{G}$, however, contradicting our assumption. □
2. Develop a recursive formula for $a_n$, which is the number of sequences for length $n \geq 1$ of elements taken from \{0, 1\} that have an odd number of 0. [Hint: Consider the 2 possibilities for choosing the first element of such a sequence.]

**Proof:** We only need to distinguish two exhaustive cases:

(a) If the first element of an odd $n$-sequence is 0, the remaining $n - 1$-sequence must have an even number of 0. Obviously, $2^{n-1} - a_{n-1}$ is the number of even $n - 1$-sequences.

(b) If the first element is 1, the remaining $n - 1$-sequence must have an odd number of 0.

So adding up the possible alternatives yields $a_n = 2^{n-1} - a_{n-1} + a_{n-1} = 2^{n-1}$. □

3. [Pre]. [SS21,Sample]. Using induction on $n \geq 1$, show that $\sum_{k=0}^{m}(-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$, for every $m \geq 0$. [Hint: Recall that $\binom{0}{0} = 1$ and $\binom{0}{1} = 0$.]

**Proof:** Basis $n = 1$: For $m = 0$, both sides are equal since $\binom{0}{0} = 1$. For $m \geq 1$, we find $1 + (-1) = 0$ as needed, since $\binom{0}{1} = 0$.

Induction step: Assume the statement holds for any $n \geq 1$ and any $m$, and show that it holds also for $n + 1$ and any $m$: Using the well-known fundamental identity $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$, we find

\[
\sum_{k=0}^{m}(-1)^k \binom{n+1}{k} = 1 + \sum_{k=1}^{m}(-1)^k \left[ \binom{n}{k} + \binom{n}{k-1} \right] = \sum_{k=0}^{m}(-1)^k \binom{n}{k} - \sum_{\ell=0}^{m-1}(-1)^\ell \binom{n}{\ell} = (-1)^m \binom{n-1}{m} - (-1)^{m-1} \binom{n-1}{m-1} = (-1)^m \binom{n}{m}.
\]

□