The Average CRI-Length of a Tree Collision Resolution Algorithm in Presence of Multiplicity-Dependent Capture Effects

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Abstract. We investigate the average length L_n of a collision resolution interval of a simple tree collision resolution algorithm in presence of capture effects: In opposition to the common model, we assume a (non-zero) probability ac^n, $c < 1$ that exactly one of the packets involved in a collision of multiplicity n survives. Our analysis is based on the application of Mellin transform techniques to ordinary generating functions, which yields an asymptotic expansion of the ordinary generating function $L(z)$ of L_n near its dominant singularity $z = 1$. The application of a simple transfer lemma eventually provides the required asymptotic expansion of L_n as $n \to \infty$.

1 Introduction

In the past, a large amount of theoretical (and practical) work has been devoted to the analysis of distributed algorithms suitable for controlling the transmission activities of stations (i.e., transmitting/receiving units) sharing a single broadcast communication channel. Applications of such algorithms may be found in certain computer networks, e.g., in wide area networks based on (satellite) radio channels. Starting from the famous ALOHA system of the Univerity of Hawaii in the late 1970's, a number of different algorithms offering much better characteristics (e.g., throughput, stability, ...) have been developed, cf. [**8**] for a nice survey.
The appropriate analysis usually relies on a model similar to the following:

(1) A (infinite) population of identical transmitters is supposed to have access to a common time-slotted, noiseless, collision-type broadcast channel, without any centralized channel arbitration mechanism.
(2) Data are transmitted in the form of fixed size packets, which fit into one time slot.
(3) For the whole population of stations, new packets are generated according to a Poisson process with a fixed rate λ.
(4) If $n \geq 2$ stations transmit in the same slot, a collision of multiplicity n occurs and all packets involved in that collision are completely lost. The channel feedback (e.g., collision/no collision) is supposed to be identical for all stations.

A very important class of collision resolution algorithms are *tree algorithms*, see [**10**] for details. We restrict ourselves to the so-called Q-ary tree algorithm with the obvious blocked access protocol, which works as follows:

No collision resolution: If the whole system is in an idle state, each transmitter having generated a new packet transmits it in the very next slot. Obviously, if there are two or more simultaneous transmission attempts, a collision occurs and a *collision resolution interval* (CRI) is initiated.

Collision resolution: Each transmitter involved in the (initial) collision flips a fair Q-sided coin with values from $1, 2, \ldots, Q$. That value determines the (relative) number of the slot where the packet should be transmitted again. For example, all stations with 1 flipped transmit in the very next slot. If a new collision occurs, it is resolved immediately by the same method, thereby defering the treatment of transmitters having other values (≥ 2) flipped. Note that such stations have to keep track with those "intermediate" resolution processes, e.g., to add Q to their previous relative slot number in order to determine the new one.
On the other hand, all transmitters not involved in the initial collision (which has terminated the idle state of the whole system) remain blocked, that is, inactive, until the whole system is idle again. They may contend for the idle slot following a collision resolution interval, probably forcing a new initial collision.

A crucial parameter for the performance of a collision resolution algorithm (of this type) is the (average) length of a collision resolution interval. This random variable denotes the number of slots necessary for resolving an initial collision of multiplicity n, and is clearly independent from the packet generating process due to the blocked access protocol.
Our intention is the derivation of the average CRI-length L_n for our simple tree algorithm when *capture effects* are present. In many real communication systems, the "strongest" of the actually colliding packets is sometimes able to capture the receiver and is therefore received without error. That is, in case of a collision there is a non-zero probability that exactly one of the packets involved survives. To handle this subject, we have to change (4) of the usual model stated above.
In [**11**], we used a very simple approach for modelling capture: In case of a collision, we assumed a fixed probability $1-p$ that exactly one packet survives. The major difficulty with this idea is the unrealistic assumption of a capture probability which is independent of the multiplicity of the collision. For example, in a radio network, the signal of a dominating packet has to be more powerful than the sum of the signals of the other packets in order to capture the receiver. Thus, the probability of capture decreases fast with the multiplicity of a collision. A suitable choice in this respect (which has the advantage of being mathematically tractable, too) is the assumption of a geometric capture probability ac^n for some $c < 1$. Obviously, $1-ac^n$ is the probability that all packets involved in a collision of multiplicity n get lost.
The outline of the paper is as follows: Section 2 contains an outline of the whole analysis and the derivation of a functional equation for the ordinary generating function of L_n, which is attacked by Mellin transform techniques in Section 3. Section 4 is devoted to some computations concerning the major term of the asymptotic expansion of L_n as $n \to \infty$ and closes with our major theorem. At last, some conclusions are appended in Section 5.

2 Functional Equations

According to [**10**], a collision resolution interval of a tree algorithm with the obvious blocked access protocol may be modelled as a Q-ary tree. Each slot is represented by a node in the tree; the root of the tree corresponds to the initial collision slot. Since the coin-flipping process splits the set of transmitters involved in a particular collision into exactly Q subsets, each node corresponding to a collision slot has exactly Q successors. Empty slots or slots where only a single transmission occurs form the leafs in our tree.

It is possible to adopt this correspondence to suit our needs. We consider Q-ary trees with two types of nodes, C-nodes (Capture-nodes, representing a successful transmission of a packet in a collision slot, or a single transmission), and NC-nodes (NonCapture-nodes, representing a collision slot with the destruction of all packets involved, or an empty slot). Each node is additionally labelled with the multiplicity of the corresponding conflict; 0 is the label for empty slots, 1 for a slot used by a single transmitter.

If we examine the trees generated by the application of these rules, we obtain the following properties:

(1) For each collision resolution interval of a conflict of multiplicity n, there exists a unique tree representation of the resolution process with exactly n C-nodes. The appropriate CRI-length L_n is reflected by the total number of nodes in the tree.

(2) Traversing the tree in preorder, we obtain the traffic on the channel; each C-node represents a slot with a successful transmission.

(3) Leafs correspond to empty slots (label 0) or to single transmission slots (label 1).

(4) Internal nodes correspond to collision slots of multiplicity equal to their label $n \geq 2$. The sum of the labels of the Q successors equals n in case of a NC-node and $n-1$ in case of a C-node.

Such trees may be viewed as a certain "mixture" of digital search trees and radix search tries, cf. [**7**] for a survey. Therefore, it is not too surprising that we succeded with an analysis based on a novel technique for investigating characteristic parameters of such trees proposed in [**6**].

We consider this approach important enough to preceed our detailled derivations with a sketch of the general idea:

Starting from the obvious recurrence relation for L_n, we derive a functional equation for the exponentially generating function (EGF) $l(z)$, namely

$$\begin{aligned} l'(z) &= (Q-1)e^{z(1-1/Q)}l(z/Q) + e^{z(1-1/Q)}l'(z/Q) \\ &\quad + ace^{zc(1-1/Q)}l(zc/Q) - ace^{zc(1-1/Q)}l'(zc/Q) + e^z - Q. \end{aligned}$$

Using $h(z) = e^{-z}l(z)$, a functional equation for $h(z)$ is easily found:

$$h(z) + h'(z) = Qh(z/Q) + h'(z/Q) - ace^{-z(1-c)}h'(zc/Q) + 1 - Qe^{-z}$$

Now, the major idea is the transition to ordinary generating functions (OGF) $H(z)$ and $L(z)$, respectively. Using the Borel-transformation (or equivalently, extracting the coefficients and summing up), we easily obtain a functional equation for $I(z) = H(z) - h_0 - h_1 z$, namely

$$I(z) = QI(z/Q) - \frac{aQ}{1+z} I\Big(\frac{zc/Q}{1+z(1-c)}\Big) + \frac{Qz^2}{(1+z)^2}.$$

Since $l(z) = e^z h(z)$ transforms into

$$L(z) = \frac{1}{1-z} H\big(\frac{z}{1-z}\big),$$

it is clear that an asymptotic expansion of $L(z)$ for $z \to 1$ may be obtained by investigating the asymptotics of $H(z)$ for $z \to \infty$. The latter is done by means of complex Mellin-transform techniques (as already used in [**5**], for example). Provided there is an (admissible) unique solution $I(z)$ of the functional equation above, it is clear that the fundamental strip of the transform of

$$T(z) = \frac{1}{1+z} I\Big(\frac{zc/Q}{1+z(1-c)}\Big)$$

is $\langle -2, 1\rangle$ due to its order as $z \to 0$ and $z \to \infty$, respectively; note that $I(z) = O(z^2)$ as $z \to 0$. Similarily, it is easily verified that the transform of $z^2/(1+z)^2$ is $\Gamma(s+2)\Gamma(-s)$, yielding the fundamental strip $\langle -2, 0\rangle$.
We therefore obtain the following equation for the transform $I^*(s) = \mathcal{M}\big[I(z); s\big]$:

$$I^*(s) = \frac{Q\Gamma(s+2)\Gamma(-s) - aQT^*(s)}{1 - Q^{1+s}}$$

The appropriate fundamental strip is $\langle -2, -1\rangle$. The "first" singularities to the right of the fundamental strip are obviously simple poles $s_k = -1 + \chi_k$ for integers k, where $\chi_k = 2k\pi i/\log Q$, caused by the vanishing denominator. The appropriate expansion reads

$$I^*(s) = -\frac{Q}{\log Q}\big(\Gamma(1+\chi_k)\Gamma(1-\chi_k) - aQT^*(-1+\chi_k)\big) \cdot \frac{1}{s+1-\chi_k} \quad \text{for } s \to -1 + \chi_k.$$

Continuing the search for further singularities, we obtain a simple pole at $s = 0$ next, which is caused by $\Gamma(s)$.
It is well-known, that such singularities conveniently translate into the asymptotic expansion of $I(z)$ on a term-by-term basis. We therefore obtain

$$I(z) = \frac{Q\big(1 - aT^*(-1)\big)}{\log Q} z + \frac{Q}{\log Q} z P(\log_Q z) + O(1)$$

for $z \to \infty$, where $P(u)$ denotes a periodic function with mean 0 and small amplitude, as usual.
Note that the term $T(z)$, which is not explicitely expressible since it depends on the unknown function $I(z)$, contributes only constant factors $T^*(-1-\chi_k)$ to the major terms of the desired asymptotics of $I(z)$ for $z \to \infty$! Actually, it is possible

to provide asymptotic expansions of $T^*(-1-\chi_k)$ as $c \to 0$ by means of a direct evaluation of the appropriate Mellin integral.
However, remembering the relation between $L(z)$ and $H(z)$, we find

$$L(z) = \frac{Q\big(1 - aT^*(-1)\big)}{\log Q} \frac{1}{(1-z)^2} + \frac{Q}{\log Q} \frac{1}{(1-z)^2} P\big(\log_Q \frac{1}{1-z}\big) + O\big(\frac{1}{1-z}\big)$$

for $z \to 1$, and the application of a simple transfer lemma finally yields the desired asymptotic expansion of L_n as $n \to \infty$.
We are convinced that this method should be applicable to other functional equations involving a certain mixture of "harmonic" arguments (e.g., $I(zQ^{-1})$) and more complicated ones (e.g., $I\big(f(z)\big)$ with $f(z) = zQ^{-1}/(1+z)$), provided that the order of the "untractable" terms guarantee a fundamental strip which is sufficiently large. It is therefore very conveniently applicable in general; however, a numerical evaluation of the appropriate contributions is not always possible.

We start our detailled derivations by providing the basic recurrence for the average total number of nodes in a tree with exactly n C-nodes, i.e., the average CRI-length L_n, $n \geq 2$:

$$L_n = 1 + (1 - ac^n) \sum_{\Sigma i_l = n} \binom{n}{i_1, \dots, i_Q} Q^{-n} \sum_{k=1}^{Q} L_{i_k}$$
$$+ ac^n \sum_{\Sigma i_l = n-1} \binom{n-1}{i_1, \dots, i_Q} Q^{1-n} \sum_{k=1}^{Q} L_{i_k}$$

This is justified by the following straightforward facts: First, the root node, i.e., the (initial) collision slot, contributes 1 to the sum above. The second term corresponds to the non-capture case, where a (root) node with label n has to have Q successors with label $i_1, \dots, i_Q$ and $\sum_k i_k = n$; note the appropriate multinomial splitting probability. Finally, the third term covers the case of an actual capture, where the label-sum of the Q successors has to be $n-1$. By virtue of the simple identity

$$\sum_{\Sigma i_l = n} \binom{n}{i_1, \dots, i_Q} Q^{-n} \sum_{k=1}^{Q} L_{i_k} = \sum_{k=1}^{Q} \sum_{i_k=0}^{n} \binom{n}{i_k} L_{i_k} Q^{-n} \sum_{\Sigma i_l = n - i_k} \binom{n - i_k}{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_Q}$$
$$= \sum_{k=1}^{Q} \sum_{i_k=0}^{n} \binom{n}{i_k} L_{i_k} Q^{-n} (Q-1)^{n-i_k}$$
$$= Q \sum_{i=0}^{n} \binom{n}{i} Q^{-i} (1 - 1/Q)^{n-i} L_i,$$

we may rewrite the initial recurrence in order to obtain

$$L_n = 1 + Q(1 - ac^n) \sum_{i=0}^{n} \binom{n}{i} Q^{-i} (1 - 1/Q)^{n-i} L_i$$
$$+ Qac^n \sum_{i=0}^{n-1} \binom{n-1}{i} Q^{-i} (1 - 1/Q)^{n-1-i} L_i \qquad \text{for } n \geq 2; \tag{2.1}$$

the initial values are $L_1 = 1$ and $L_0 = 1$.
It is convenient to proceed with some well-known techniques from the formal operator calculus. Let $\mathcal{D}$ denote the ordinary differential operator, $\mathcal{I}$ the identity operator and $\mathcal{N}$ the 0-substitution operator, all with respect to a variable t. Now, if $l(t)$ denotes the *exponential generating function* (EGF) of L_n, it is clear that $L_n = \mathcal{N}\mathcal{D}^n * l(t)$; the star characterizes the application of the operator(s) on the left hand side to the function on the right.
Rewriting our recurrence relation (2.1) accordingly, we obtain for $n \geq 2$

$$L_n = 1 + Q(1-ac^n)\mathcal{N}(1-1/Q+\mathcal{D}/Q)^n * l(t) + Qac^n\mathcal{N}(1-1/Q+\mathcal{D}/Q)^{n-1} * l(t).$$

Multiplying both sides by $z^{n-1}/(n-1)!$ and summing up for $n \geq 2$, we find by using the fact $\mathcal{N}e^{x\mathcal{D}} * f(t) = f(x)$

$$\begin{aligned}
l'(z) - L_1 &= e^z - 1 + Q\mathcal{N}(1-1/Q+\mathcal{D}/Q)\left[e^{z(1-1/Q+\mathcal{D}/Q)} - \mathcal{I}\right] * l(t)\\
&\quad - Qac\mathcal{N}(1-1/Q+\mathcal{D}/Q)\left[e^{zc(1-1/Q+\mathcal{D}/Q)} - \mathcal{I}\right] * l(t)\\
&\quad + Qac\mathcal{N}\left[e^{zc(1-1/Q+\mathcal{D}/Q)} - \mathcal{I}\right] * l(t)\\
&= e^z - 1 + (Q-1)e^{z(1-1/Q)}l(z/Q) - (Q-1)L_0 + e^{z(1-1/Q)}l'(z/Q) - L_1\\
&\quad - ac(Q-1)e^{zc(1-1/Q)}l(zc/Q) + ac(Q-1)L_0\\
&\quad - ace^{zc(1-1/Q)}l'(zc/Q) + acL_1\\
&\quad + Qace^{zc(1-1/Q)}l(zc/Q) - QacL_0\\
&= (Q-1)e^{z(1-1/Q)}l(z/Q) + e^{z(1-1/Q)}l'(z/Q)\\
&\quad + ace^{zc(1-1/Q)}l(zc/Q) - ace^{zc(1-1/Q)}l'(zc/Q)\\
&\quad + e^z - 1 - (Q-1)L_0 - L_1 - acL_0 + acL_1
\end{aligned}$$

and therefore

$$\begin{aligned}
l'(z) &= (Q-1)e^{z(1-1/Q)}l(z/Q) + e^{z(1-1/Q)}l'(z/Q)\\
&\quad + ace^{zc(1-1/Q)}l(zc/Q) - ace^{zc(1-1/Q)}l'(zc/Q) + e^z - Q.
\end{aligned} \tag{2.2}$$

Introducing the *Poisson generating function* (PoGF) of L_n, namely

$$h(z) = \sum_{n\geq 0} h_n \frac{z^n}{n!} = l(z)e^{-z}, \tag{2.3}$$

which implies the inverse pair

$$h_n = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}L_k \qquad \text{and} \qquad L_n = \sum_{k=0}^{n}\binom{n}{k}h_k, \tag{2.4}$$

we have $h_0 = 1$, $h_1 = 0$ and $e^{-z}l'(z) = h(z) + h'(z)$. Thus, multiplying equation (2.2) by e^{-z} yields a simpler functional equation for $h(z)$:

$$\begin{aligned}
h(z) + h'(z) &= (Q-1)h(z/Q) + h(z/Q) + h'(z/Q) - ace^{z(c-1)}h'(zc/Q) + 1 - Qe^{-z}\\
&= Qh(z/Q) + h'(z/Q) - ace^{-z(1-c)}h'(zc/Q) + 1 - Qe^{-z}
\end{aligned} \tag{2.5}$$

The major idea for treating this equation, as proposed in [**6**], is the transition to *ordinary generating functions* (OGFs). If

$$L(z) = \sum_{n\geq 0} L_n z^n \qquad \text{and} \qquad H(z) = \sum_{n\geq 0} h_n z^n$$

denote the appropriate OGFs, (2.4) implies the following relation:

$$\begin{aligned} L(z) &= \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k} h_k z^n = \sum_{k\geq 0} h_k \sum_{n\geq k}\binom{n}{k} z^n = \sum_{k\geq 0} h_k z^k \sum_{n\geq 0}\binom{n+k}{k} z^n \\ &= \frac{1}{1-z} H\big(\frac{z}{1-z}\big) \end{aligned} \tag{2.6}$$

Translating the functional equation (2.5) for the EGF into the corresponding one for the OGF is simple. We use the Borel-transformation in a more or less "formal" manner; alternatively, one might extract the coefficients and sum up as well. Providing

$$\int_0^\infty e^{-t} h'(\alpha t)\, dt = -h_0/\alpha + \frac{1}{\alpha}\int_0^\infty e^{-t} h(\alpha t)\, dt = \frac{H(\alpha) - h_0}{\alpha}$$

and

$$\int_0^\infty e^{-t} e^{-\beta t} h'(\alpha t)\, dt = \frac{1}{1+\beta}\int_0^\infty e^{-s} h'\big(\frac{\alpha s}{1+\beta}\big)\, ds = \frac{H\big(\frac{\alpha}{1+\beta}\big) - h_0}{\alpha},$$

we obtain by substituting $z = zt$ in (2.5), multiplying by e^{-t} and integrating

$$H(z) + \frac{H(z) - h_0}{z} = QH(z/Q) + \frac{H(z/Q) - h_0}{z/Q} - ac\frac{H\big(\frac{zc/Q}{1+z(1-c)}\big) - h_0}{zc/Q} + 1 - \frac{Q}{1+z}.$$

Introducing

$$I(z) = H(z) - h_1 z - h_0 = H(z) - 1, \tag{2.7}$$

cf. equations (2.4) and (2.1), we find

$$I(z)(1 + 1/z) = QI(z/Q)(1 + 1/z) + Q - \frac{aQ}{z} I\Big(\frac{zc/Q}{1+z(1-c)}\Big) - \frac{Q}{1+z};$$

some straightforward algebraic manipulations finally yield

$$I(z) = QI(z/Q) - \frac{aQ}{1+z} I\Big(\frac{zc/Q}{1+z(1-c)}\Big) + \frac{Qz^2}{(1+z)^2}. \tag{2.8}$$

In order to show that the functional equation (2.8) has indeed a suitable analytic solution, it is possible to employ similar techniques as in [**3**] to prove the following

LEMMA 2.1 (EXISTENCE OF AN ANALYTIC SOLUTION). *Let $t(z)$, $\gamma_1(z)$, $\gamma_2(z)$, $\sigma_1(z)$, $\sigma_2(z)$ denote analytic functions defined on a (possibly open) domain $\mathcal{D}$ of the complex plane. Let H be the semigroup of substitutions generated by σ_1 and σ_2 under the operation of composition of functions: The identity of H is $\varepsilon = \varepsilon(z) = z$; any member $\sigma = \sigma(z) \in H$ can be written in the form $\sigma = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_n}$, $|\sigma| = n \geq 0$ and $i_k \in \{1,2\}$ for $1 \leq k \leq n$.*
If (1) $\sigma_1(z)$ and $\sigma_2(z)$ are such that $\sigma_i(\mathcal{D}) \subseteq \mathcal{D}$ (and therefore $\tau\sigma(\mathcal{D}) \subseteq \tau(\mathcal{D})$ for any $\tau, \sigma \in H$) and $|\sigma_i(z)| \leq |z|$ for $i = 1, 2$, and (2) for some $m \geq 0$ arbitrary but fixed

$$\alpha_1 = \sup_{\substack{z\in\mathcal{D}\\|\sigma|=m}} \left|\gamma_1\big(\sigma(z)\big)\right|, \qquad \alpha_2 = \sup_{\substack{z\in\mathcal{D}\\|\sigma|=m}} \left|\gamma_2\big(\sigma(z)\big)\right|$$

satisfy the contraction property $\alpha_1 + \alpha_2 < 1$, then the functional equation

$$J(z) = t(z) + \gamma_1(z)J\big(\sigma_1(z)\big) + \gamma_2(z)J\big(\sigma_2(z)\big)$$

has an analytic solution

$$J(z) = \sum_{\sigma\in H} [\gamma_1, \gamma_2]^\sigma(z)\, t\big(\sigma(z)\big)$$

on $\mathcal{D}$, where

$$[\gamma_1,\gamma_2]^\sigma(z) = \gamma_{i_1}\big(\sigma_{i_2}\ldots\sigma_{i_n}(z)\big)\ \cdots\ \gamma_{i_{n-1}}\big(\sigma_{i_n}(z)\big)\cdot\gamma_{i_n}(z) \quad \text{for } |\sigma| = n,\ n \geq 1,$$
$$[\gamma_1,\gamma_2]^\varepsilon(z) = 1.$$

Proof: Since H satisfies the decomposition $H = \{\varepsilon\} \cup H\sigma_1 \cup H\sigma_2$, we easily obtain that

$$\begin{aligned}
J(z) &= \sum_{\sigma\in H} [\gamma_1,\gamma_2]^\sigma(z)\, t\big(\sigma(z)\big)\\
&= t(z) + \sum_{\substack{\sigma=\tau\sigma_1\\ \tau\in H}} [\gamma_1,\gamma_2]^\tau\big(\sigma_1(z)\big)\gamma_1(z)\, t\big(\tau\sigma_1(z)\big)\\
&\qquad + \sum_{\substack{\sigma=\tau\sigma_2\\ \tau\in H}} [\gamma_1,\gamma_2]^\tau\big(\sigma_2(z)\big)\gamma_2(z)\, t\big(\tau\sigma_2(z)\big)\\
&= t(z) + \gamma_1(z)\sum_{\tau\in H}[\gamma_1,\gamma_2]^\tau\big(\sigma_1(z)\big)t\big(\tau\big(\sigma_1(z)\big)\big)\\
&\qquad + \gamma_2(z)\sum_{\tau\in H}[\gamma_1,\gamma_2]^\tau\big(\sigma_2(z)\big)t\big(\tau\big(\sigma_2(z)\big)\big)\\
&= t(z) + \gamma_1(z)J\big(\sigma_1(z)\big) + \gamma_2(z)J\big(\sigma_2(z)\big)
\end{aligned}$$

is actually a (formal) solution of our functional equation. Defining

$$\mu(z) = \sup_{\substack{|w|\leq|z|\\ w\in\mathcal{D}}} \left|t(w)\right|$$

$$\alpha_1(z) = \sup_{\substack{|w|\leq|z|\\ w\in\mathcal{D}}} |\gamma_1(w)| \qquad \text{and} \qquad \alpha_2(z) = \sup_{\substack{|w|\leq|z|\\ w\in\mathcal{D}}} |\gamma_2(w)|$$

for any $z \in \mathcal{D}$ with bounded modulus, condition (1) ensures that $\left|t(\sigma(z))\right| \leq \mu(z)$ and $\left|\gamma_i(\sigma(z))\right| \leq \alpha_i(z)$, $i = 1, 2$, for any $\sigma \in H$. Moreover, for any $\sigma \in H$ with $|\sigma| = l$ we have

$$\left|[\gamma_1, \gamma_2]^\sigma(z)\right| \leq \alpha_{i_1} \cdots \alpha_{i_{l-m}} \cdot \alpha_{i_{l-m+1}}(z) \cdots \alpha_{i_l}(z) = \alpha_{i_1} \cdots \alpha_{i_l} \cdot \frac{\alpha_{i_{l-m+1}}(z)}{\alpha_{i_{l-m+1}}} \cdots \frac{\alpha_{i_l}(z)}{\alpha_{i_l}}.$$

Denoting $M(z) = \max\{1, \alpha_1(z)/\alpha_1, \alpha_2(z)/\alpha_2\}$, we eventually find

$$\begin{aligned}\sum_{\sigma \in H} \left|[\gamma_1, \gamma_2]^\sigma(z)\, t(\sigma(z))\right| &\leq \mu(z) \sum_{\sigma \in H} \left|[\gamma_1, \gamma_2]^\sigma(z)\right| \\ &\leq \mu(z) M(z)^m \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k} \alpha_1^k \alpha_2^{n-k} \\ &= \frac{\mu(z) M(z)^m}{1 - \alpha_1 - \alpha_2},\end{aligned}$$

which reveals that our solution is represented by a uniformly convergent sum of analytic functions and is hence analytic itself. This completes the proof of Lemma 2.1. ∎

Lemma 2.1 is not directly applicable to the functional equation (2.8) since the required contraction property is violated. However, remembering definition (2.7), it is obvious that $I(z) = z^2 J(z)$. Rewriting (2.8) accordingly, we obtain

$$J(z) = Q^{-1} J(z/Q) - \frac{ac^2/Q}{(1+z)\big(1+z(1-c)\big)^2} J\Big(\frac{zc/Q}{1+z(1-c)}\Big) + \frac{Q}{(1+z)^2}.$$

It is easily checked that this equation is treatable by Lemma 2.1 on

$$\mathcal{D} = \{z : |\operatorname{Arg}(z + 2/3)| \leq \psi\} \cap \{z : |\operatorname{Arg}(z+1)| \leq \phi\} \tag{2.9}$$

for some $\psi > \phi > \pi/2$; the reason for that (somewhat artificial) region will become clear in the following section. Note that $\gamma_2(z) < 1/Q$ ($Q \geq 2$) is only valid for c small enough ($\gamma_2(z)$ attains its maximum on $\mathcal{D}$ at $z = -2/3$). However, using the "feature" $m \geq 1$ in our Lemma, it is possible to handle arbitrary $c < 1$ as well.

Thus, Lemma 2.1 states the existence of a solution $J(z)$ (and hence $I(z)$), which is analytic on $\mathcal{D}$. Moreover, the fact that the coefficients of $I(z)$ are uniquely defined (which is immediately apperarent from the functional equation (2.5); cf. the recurrence relation (4.4), too) implies that this solution is uniquely determined.

In addition, by a simple majorization of (2.8) divided by z,

$$\frac{I(z)}{z} = \frac{I(z/Q)}{z/Q} + O(1/z) \qquad \text{for } z \to +\infty,$$

it is easily seen that

$$I(z) = O(z) \qquad \text{for } z \to +\infty, \tag{2.10}$$

i.e., that the order of $I(z)$ for $z \to \infty$ is smaller than its order as $z \to 0$.

3 Mellin Transform Techniques

As we shall see, equation (2.8) is tractable by Mellin transform techniques. The well-known Mellin transform is a powerful tool in asymptotic analysis, and is applicable to a wide variety of problems; see [**2**] for the very complete theory and [**12**] for application-oriented details. First, we supply a short summary of appropriate theorems (without any proof):

Definition 3.1 (Mellin-Transform). *The Mellin-transform of a continuous, real valued function $f(x)$ is the complex valued function $f^*(s)$ defined by*

$$f^*(s) = \mathcal{M}[f(x); s] = \int_0^\infty f(x) x^{s-1}\, dx,$$

provided that the integral is absolutely convergent in the region $\alpha < \Re(s) < \beta$. This region is called the fundamental strip of $f^*(s)$ and is denoted by $\langle \alpha, \beta \rangle$.

Theorem 3.2 (Existence and Analyticity). *If there are two complex numbers α, β with $\Re(-\alpha) < \Re(-\beta)$ and the property*

$$f(x) = \begin{cases} O(x^\alpha), & \text{for } x \to 0 \\ O(x^\beta), & \text{for } x \to \infty, \end{cases}$$

e.g., if the order of $f(x)$ near zero is larger than the order near infinity, then the transform $f^*(s)$ exists on the fundamental strip $\langle -\alpha, -\beta \rangle$, and is analytic within the whole region.

Lemma 3.3 (Transform of Harmonic Functions). *If the transform $f^*(s)$ of the function $f(x)$ exists on the fundamental strip $\langle \alpha, \beta \rangle$, we obtain for a real $c > 0$*

$$\mathcal{M}[f(cx); s] = c^{-s} f^*(s) \qquad \text{within the fundamental strip } \langle \alpha, \beta \rangle.$$

Theorem 3.4 (Asymptotic Expansion). *If $f^*(s)$ exists on the fundamental strip $\langle \alpha, \beta \rangle$, and satisfies certain smallness-conditions towards $i\infty$ for $\beta \le \Re(s) \le M$ (to the right of the fundamental strip), a pole of the transform*

$$f^*(s) \sim \sum_{n=0}^{N} \frac{d_{n,k}}{(s - b_k)^{n+1}} \qquad \text{for } \beta \le \Re(b_k) < M \text{ and } s \to b_k$$

translates into a term of the asymptotic expansion for $x \to \infty$ according to

$$\sum_{n=0}^{N} \frac{-d_{n,k}}{n!} (-\log x)^n x^{-b_k} \qquad \text{for } x \to \infty.$$

Moreover, the complete expansion of $f(x)$ evaluates to

$$f(x) = \sum_k \text{Term resulting from } b_k + O(x^{-M}) \qquad \text{for } x \to \infty.$$

The last theorem is a consequence of the classical inversion theorem

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s}\, ds,$$

where c lies in the fundamental strip of $f^*(s)$. The idea behind the proof of Theorem 3.4 is to extend the contour by a large rectangular one in the right halfplane, and

to take into account the residues of the newly enclosed singularities. The requested smallness-conditions ensure vanishing contributions resulting from the horizontal segments when expanding the contour to $\pm i\infty$. Note that the smallness-conditions are always satisfied within the fundamental strip.
The classical theory of the Mellin transform is based on real functions $f(x)$. However, our theorems (especially Theorem 3.4) are valid for (certain) complex functions $f(z)$ as well. There are a number of strikingly elegant applications (cf. [**5**] or [**6**], for example) which demonstrate the power of this approach. We, too, shall use this technique to attack the functional equation (2.8).
First of all, equations (2.7) and (2.10) in conjunction with Theorem 3.2 guarantee that the Mellin transform $I^*(s)$ of our desired solution $I(z)$ exists. Abbreviating

$$T(z) = \frac{1}{1+z} I\Big(\frac{zc/Q}{1+z(1-c)}\Big) = \begin{cases} O(z^2) & \text{for } z \to 0, \\ O(z^{-1}) & \text{for } z \to \infty; \end{cases}$$

the appropriate fundamental strip of $T^*(s)$ is $\langle -2, 1\rangle$ by virtue of Theorem (3.2). Regarding the last term of (2.8), we have

$$\int_0^\infty \frac{z^{s+1}}{(1+z)^2}\, dz = B(s+2, -s) = \Gamma(s+2)\Gamma(-s)$$

according to [**1**, p.258]; $B(z,w)$ denotes the Beta function. Since $\Gamma(s+2)\Gamma(-s)$ is analytic within the strip $-2 < \Re(s) < 0$, we obtain the fundamental strip $\langle -2, 0\rangle$. Now, using Theorem (3.3), the Mellin transform of $I(z)$ evaluates to

$$I^*(s) = \frac{Q\Gamma(s+2)\Gamma(-s) - aQT^*(s)}{1-Q^{1+s}}; \tag{3.1}$$

the appropriate fundamental strip is obviously $\langle -2, -1\rangle$.
According to Theorem (3.4), the desired asymptotics for $z \to \infty$ are trivially connected with the singularities of the transform to the right of the fundamental strip. The "first" singularities encountered are simple poles at $-1+\chi_k$, $\chi_k = 2k\pi i/\log Q$ for all integers k, which are caused by the vanishing denominator. Applying de l'Hospital's rule, we find

$$\frac{1}{1-Q^{1+s}} = -\frac{1}{\log Q} \cdot \frac{1}{s+1-\chi_k} + O(1) \qquad \text{for } s \to -1+\chi_k$$

and therefore

$$I^*(s) = -\frac{Q}{\log Q}\big(\Gamma(1+\chi_k)\Gamma(1-\chi_k) - aT^*(-1+\chi_k)\big) \cdot \frac{1}{s+1-\chi_k} \qquad \text{for } s \to -1+\chi_k.$$

The next singularity is a simple pole at $s = 0$, caused by $\Gamma(s)$. The expansion is simple:

$$I^*(s) = -\frac{Q}{Q-1} \cdot \frac{1}{s} + O(1) \qquad \text{for } s \to 0.$$

Further singularities lie on the vertical strip $\Re(s) = 1$. But, since we have to expect a singularity caused by the (not explicitely known) function $T^*(s)$, there is no easy way

to state the appropriate expansion. However, to obtain an asymptotic expansion of $I(z)$ up to $O(z^{-1+\varepsilon})$ for $z \to \infty$, it suffices that there are no additional singularities within the strip $0 < \Re(s) \le 1-\varepsilon$ for some $\varepsilon > 0$.
It is not hard to show that $I^*(s)$ is indeed small towards $\pm i\infty$ within the strip $-3/2 \le \Re(s) \le 1-\varepsilon$, as required by Theorem 3.4: $I^*(s)$ involves $\Gamma(s)$, which is well-known to be small for large imaginary parts of s, and $T^*(s)$, whose fundamental strip covers the region of interest (because smallness-conditions are always satisfied within the fundamental strip).
Therefore, we may apply Theorem (3.4) and obtain

$$I(z) = \frac{Q\big(1-aT^*(-1)\big)}{\log Q}\cdot z + \frac{Q}{\log Q}\cdot zP(\log_Q z) + \frac{Q}{Q-1} + O(z^{-1+\varepsilon}) \qquad (3.2)$$

for $z \to \infty$; $\varepsilon > 0$ denotes an arbitrary small positive constant, and

$$P(u) = \sum_{k\neq 0}\big(\Gamma(1+\chi_k)\Gamma(1-\chi_k) - aT^*(-1+\chi_k)\big)e^{-2k\pi i u}$$

is a periodic function with period 1, mean 0 and small amplitude, caused by poles with non-zero imaginary part.
Note that the term $T(z)$, which is not explicitely expressible since it depends on the unknown function $I(z)$, contributes a constant factor to the major term of the desired asymptotics of $I(z)$ for $z \to \infty$! We shall provide an asymptotic expansion of this factor as $c \to 0$ in Section 4.

However, using relation (2.6) and (a weaker form of) expansion (3.2) yield

$$\begin{aligned} L(z) &= \frac{1}{1-z} I\Big(\frac{z}{1-z}\Big) + \frac{1}{1-z} \\ &= \frac{Q\big(1-aT^*(-1)\big)}{\log Q}\frac{1}{(1-z)^2} + \frac{Q}{\log Q}\frac{1}{(1-z)^2}P\Big(\log_Q \frac{z}{1-z}\Big) + O\Big(\frac{1}{1-z}\Big) \end{aligned}$$

for $z \to 1$. In addition, since $z^{-\chi_k} = 1 + O(z-1)$ for $z \to 1$ uniformly in k and therefore $P\big(\log_Q \frac{z}{1-z}\big) = P\big(\log_Q \frac{1}{1-z}\big) + O(z-1)$, we eventually find

$$L(z) = \frac{Q\big(1-aT^*(-1)\big)}{\log Q}\frac{1}{(1-z)^2} + \frac{Q}{\log Q}\frac{1}{(1-z)^2}P\Big(\log_Q \frac{1}{1-z}\Big) + O\Big(\frac{1}{1-z}\Big)$$

for $z \to 1$.
The desired coefficient $L_n = [z^n]L(z)$ is most easily obtained by a simple transfer lemma, cf. [4] for a very complete treatment of the subject. For applicability, we need the analyticity of $L(z)$ in a closed domain $\Delta(\phi,\eta) = \big\{z : |z| \le 1+\eta, |\operatorname{Arg}(z-1)| \ge \phi, z \ne 1\big\}$ (an indented disk with radius $1+\eta$, sparing out the singularity $z = 1$). This, however, is a straightforward consequence of Relation (2.6) and the region of analyticity of $I(z)$ according to equation (2.9), as may be seen by considering the image of $\Delta(\phi,\eta)$ under the mapping $z/(1-z)$.
Thus, we obtain

$$L_n = \frac{Q\big(1-aT^*(-1)\big)}{\log Q} n + \frac{Q}{\log Q} np(\log_Q n) + O(1) \qquad \text{for } n \to \infty, \qquad (3.3)$$

where

$$\begin{aligned} p(u) &= \sum_{k\neq 0} \frac{\Gamma(1+\chi_k)\Gamma(1-\chi_k) - aT^*(-1+\chi_k)}{\Gamma(2-\chi_k)} e^{-2k\pi i u} \\ &= \sum_{k\neq 0} \frac{\chi_k(-1+\chi_k)\Gamma(-1+\chi_k)\Gamma(1-\chi_k) - aT^*(-1+\chi_k)}{(1-\chi_k)\Gamma(1-\chi_k)} e^{-2k\pi i u} \\ &= \sum_{k\neq 0} \big(\chi_k\Gamma(-1-\chi_k) - \frac{aT^*(-1-\chi_k)}{(1+\chi_k)\Gamma(1+\chi_k)}\big) e^{2k\pi i u}. \end{aligned}$$

Note that we restricted ourselves to the computation of the major terms of the asymptotic expansion for $L(z)$ and hence L_n in order to preserve the essentials. Nevertheless, it would be easy to extend the derivations above to obtain a more accurate expression up to a remainder term of order $O(n^{-(1-\varepsilon)})$.
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4 Evaluation of a Mellin Integral

At last, we shall derive an asymptotic expression for $T^*(-1)$ as $c \to 0$, thereby providing a means for computing the major term of L_n numerically. Our point of application is the Mellin integral

$$T^*(-1) = \int_0^\infty I\Big(\frac{zc/Q}{1+z(1-c)}\Big)\frac{z^{-2}}{1+z}\,dz. \tag{4.1}$$

The fact that the argument of $I(.)$ in the integrand above remains small, i.e., $O(c)$, over the whole path of integration, enables us to operate with the first few terms of the Taylor expansion

$$I(z) = H(z) - h_0 - h_1 z = \sum_{k\geq 0} h_{k+2} z^{k+2}$$

instead of $I(z)$; remember definition (2.7). That is, using the substitution

$$t = \frac{zc}{1+z(1-c)},$$

such that

$$z = \frac{t}{c-t(1-c)} \qquad \text{and} \qquad dz = \frac{c\,dt}{\big(c-t(1-c)\big)^2} = cz^2/t^2\,dt,$$

we obtain

$$\begin{aligned} T^*(-1) &= \int_0^{\frac{c}{1-c}} I(t/Q)\frac{c-t(1-c)}{c+tc}\frac{c\,dt}{t^2} = cQ^{-2}\int_0^{\frac{c}{1-c}} \frac{I(t/Q)}{(t/Q)^2}\frac{1+t-t/c}{1+t}\,dt \\ &= cQ^{-2}\sum_{k\geq 0} h_{k+2}Q^{-k}\int_0^{\frac{c}{1-c}} \frac{1+t-t/c}{1+t}t^k\,dt. \end{aligned}$$

Since it is not hard to establish that

$$\int_0^{\frac{c}{1-c}} \frac{t^k}{1+t}\,dt = \int_0^c v\Big(\frac{v}{1-v}\Big)^{k-1}\frac{dv}{(1-v)^2} = \int_0^c \frac{v^k}{(1-v)^{k+1}}\,dv$$
$$= \sum_{m\ge 0}\binom{m+k}{m}\frac{c^{m+k+1}}{m+k+1}$$

and therefore

$$\int_0^{\frac{c}{1-c}} \frac{1+t-t/c}{1+t}t^k\,dt = \frac{1}{k+1}\Big(\frac{c}{1-c}\Big)^{k+1} - \frac{1}{c}\int_0^{\frac{c}{1-c}}\frac{t^{k+1}}{1+t}\,dt$$
$$= \sum_{m\ge 0}\bigg[\binom{m+k}{m}\frac{c^{m+k+1}}{k+1} - \binom{m+k+1}{m}\frac{c^{m+k+1}}{m+k+2}\bigg]$$
$$= \sum_{m\ge 0}\binom{m+k}{m}\frac{c^{m+k+1}}{k+1}\bigg[1-\frac{m+k+1}{m+k+2}\bigg]$$
$$= \sum_{m\ge 0}\binom{m+k}{m}\frac{c^{m+k+1}}{(k+1)(m+k+2)},$$

some algebraic manipulations eventually yield

$$T^*(-1) = cQ^{-2}\sum_{k\ge 0} h_{k+2}Q^{-k}\sum_{m\ge 0}\binom{m+k}{m}\frac{c^{m+k+1}}{(k+1)(m+k+2)}$$
$$= cQ^{-2}\sum_{l\ge 0}\sum_{k\le l} h_{k+2}Q^{-k}\binom{l}{k}\frac{c^{l+1}}{(k+1)(l+2)}$$
$$= cQ^{-2}\sum_{l\ge 0}\frac{c^{l+1}}{(l+2)}\sum_{k\le l}\binom{l}{k}h_{k+2}\frac{Q^{-k}}{k+1}$$
$$= cQ^{-2}\sum_{l\ge 0}\frac{c^{l+1}}{(l+2)(l+1)}\sum_{k\le l}\binom{l+1}{k+1}h_{k+2}Q^{-k}$$
$$= cQ^{-1}\sum_{l\ge 1}\frac{c^{l}}{l(l+1)}\sum_{k=1}^{l}\binom{l}{k}h_{k+1}Q^{-k}. \qquad (4.2)$$

What remains to do is to compute the first few h_k. This is accomplished by extracting the coefficients from the functional equation (2.5): For $n\ge 1$, we find

$$h_n + h_{n+1} = h_nQ^{-(n-1)} + h_{n+1}Q^{-n} - ac\sum_{k=0}^{n}\binom{n}{k}\big(-(1-c)\big)^{n-k}h_{k+1}(c/Q)^k - Q(-1)^n$$

and hence

$$h_{n+1}(1-Q^{-n}) + h_n(1-Q^{-(n-1)}) =$$
$$-Q(-1)^n - ac(-1)^n\sum_{k=0}^{n}\binom{n}{k}h_{k+1}(-c/Q)^k(1-c)^{n-k}.$$

Multiplying the equation above with $(-1)^{m-n}$ and summing up for $1 \le n \le m$, $m \ge 1$ while remembering $h_1 = 0$ yields a telescoping sum on the left hand side and therefore

$$
\begin{aligned}
h_{m+1}(1-Q^{-m}) &= -mQ(-1)^m - ac\sum_{n=1}^{m}(-1)^m \sum_{k=1}^{n}\binom{n}{k} h_{k+1}(-c/Q)^k(1-c)^{n-k} \\
&= mQ(-1)^{m+1} + ac(-1)^{m+1}\sum_{k=1}^{m} h_{k+1}(-c/Q)^k \sum_{n=k}^{m}\binom{n}{k}(1-c)^{n-k} \\
&= mQ(-1)^{m+1} + ac(-1)^{m+1}\sum_{k=1}^{m} h_{k+1}(-c/Q)^k t_{m,k}(c). \qquad (4.3)
\end{aligned}
$$

Evaluating $t_{m,k}(c)$ involves some standard identities concerning binomial coefficients and yields

$$
\begin{aligned}
t_{m,k}(c) &= \sum_{n=0}^{m-k}\binom{n+k}{k}(1-c)^n = \sum_{n=0}^{m-k}\binom{n+k}{n}\sum_{j=0}^{n}\binom{n}{j}(-c)^j \\
&= \sum_{j=0}^{m-k}(-c)^j \sum_{n=j}^{m-k}\binom{n+k}{n}\binom{n}{j} = \sum_{j=0}^{m-k}(-c)^j \sum_{n=j}^{m-k}\binom{n+k}{n-j}\binom{k+j}{j} \\
&= \sum_{j=0}^{m-k}(-c)^j \binom{k+j}{j}\sum_{n=0}^{m-k-j}\binom{n+k+j}{n} \\
&= \sum_{j=0}^{m-k}(-c)^j \binom{k+j}{j}\binom{m+1}{m-k-j}.
\end{aligned}
$$

Using $t_{m,m}(c) \equiv 1$ for all $m \ge 1$, some straightforward algebraic manipulations of equation (4.3) establish

$$
h_{m+1}\big(1-Q^{-m}+ac^{m+1}Q^{-m}t_{m,m}(c)\big) = mQ(-1)^{m+1} + ac(-1)^{m+1}\sum_{k=1}^{m-1} h_{k+1}(-c/Q)^k t_{m,k}(c)
$$

and therefore

$$
h_{m+1} = (-1)^{m+1}\frac{mQ - aQ\sum_{k=2}^{m} h_k(-c/Q)^k t_{m,k-1}(c)}{1-Q^{-m}(1-ac^{m+1})}. \qquad (4.4)
$$

For example, we most easily get

$$
\begin{aligned}
h_2 &= \frac{Q}{1-Q^{-1}(1-ac^2)} = \frac{Q}{1-Q^{-1}} + O(c^2) \\
h_3 &= -\frac{2Q+O(c^2)}{1-Q^{-2}(1-ac^3)} = -\frac{2Q}{1-Q^{-2}} + O(c^2);
\end{aligned}
$$

for $c \to 0$. Remembering equation (3.3) and using the above values in equation (4.2) finally establishes

$$\begin{aligned} T^*(-1) &= cQ^{-1}\left[\frac{ch_2Q^{-1}}{2} + \frac{c^2(2h_2Q^{-1}+h_3Q^{-2})}{6} + O(c^3)\right] \\ &= h_2\Big(\frac{c^2}{2Q^2} + \frac{c^3}{3Q^2}\Big) + h_3\frac{c^3}{6Q^3} + O(c^4) \\ &= \frac{c^2}{2Q(1-Q^{-1})} + \frac{c^3}{3Q(1-Q^{-1})} - \frac{c^3}{3Q^2(1-Q^{-2})} + O(c^4). \end{aligned}$$

and hence our major result

THEOREM 4.1. *The average CRI-length L_n of the Q-ary tree algorithm with obvious blocked access protocol in presence of multiplicity-dependent capture with probability ac^n is*

$$L_n = \frac{Q\big(1-aT^*(-1)\big)}{\log Q}n + \frac{Q}{\log Q}np(\log_Q n) + O(1) \qquad \text{for } n \to \infty.$$

The constant $T^*(-1)$ is approximated by

$$T^*(-1) = \frac{c^2}{2Q(1-Q^{-1})} + \frac{c^3}{3Q(1-Q^{-1})} - \frac{c^3}{3Q^2(1-Q^{-2})} + O(c^4) \qquad \text{for } c \to 0,$$

the function $p(u)$ is periodic with periode 1, has very low amplitude, mean 0, and its Fourier expansion is given by

$$p(u) = \sum_{k\neq 0}\Big(\chi_k\Gamma(-1-\chi_k) - \frac{aT^*(-1-\chi_k)}{(1+\chi_k)\Gamma(1+\chi_k)}\Big)e^{2k\pi i u} \qquad \text{with } \chi_k = \frac{2k\pi i}{\log Q}.$$

5 Conclusions

Using Mellin transforms of ordinary generating functions in conjunction with transfer lemmas, we analyzed the average CRI-length of a Q-ary tree algorithm in presence of multiplicity-dependent capture effects. Relying on a geometric capture probability of ac^n, $c < 1$ in case of a collision of multiplicity n, our investigations extend the somewhat unrealistic assumption of a constant probability $a = 1 - p$ (i.e., $c = 1$) pursued in [**11**].
However, we should admit that even our improved approach is not able to cover all possible varieties of capture effects sufficiently. For instance, in communication networks without a central receiver, capture is often a local phenomenon, i.e., concerns not all receivers in the same way. Our assumption of all stations agreeing on the channel feedback is therefore too optimistic. The consideration of local phenomenons, however, is not at all simple and requires a substantial redesign of the model and the whole analysis. æ

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