A Brief Introduction to Control Theory

Harald Paulitsch
Classification of Control Systems (2)

- $w(t)$
- $e(t) = w(t) - x(t)$
- Controller
- $u(t)$
- Control path
- $z(t)$
- Measurement device
- $x(t)$
Classification of Control Systems

• Open-loop control systems
  – control value only depends on input signal

• Closed-loop control systems
  – with feedback loop
  ⇒ also called “feedback control systems”
Advantages of Feedback Control Systems

• System output can be made to automatically follow the temporal value of the input function.
• The system performance is less sensitive to variations of parameter values.
• The system performance is less sensitive to unwanted disturbances.
Disadvantages of Feedback Control Systems

- Potential of instability (stability is a major design concern).
- Additional components for providing the feedback signal are required (measurement).
Laplace Transformation (1)

Calculation in the Laplace domain (s).

\[ s = \sigma + j\omega \]

\[ X(s) = L\{x(t)\} = \int_{0}^{\infty} x(t) \cdot e^{-st} \, dt \]

\[ x(t) = L^{-1}\{X(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \cdot e^{st} \, ds \]
Laplace Transformation (2)

Important Laplace Transformations

<table>
<thead>
<tr>
<th>x(t)</th>
<th>X(t) = L{x(t)}</th>
</tr>
</thead>
<tbody>
<tr>
<td>δ(t)</td>
<td>1</td>
</tr>
<tr>
<td>1 or σ(t)</td>
<td>1 / s</td>
</tr>
<tr>
<td>t</td>
<td>1 / s²</td>
</tr>
<tr>
<td>1/n! tⁿ</td>
<td>1 / sⁿ⁺¹</td>
</tr>
<tr>
<td>eᵃ·t</td>
<td>1 / (s – a)</td>
</tr>
</tbody>
</table>
The Transfer Function

• The transfer function $G(s)$ is defined as the ratio of the Laplace transformed output signal $Y(s)$ to the input signal $X(s)$:

$$G(s) = \frac{Y(s)}{X(s)}$$

• In the time domain, the transfer function is a map

$$f(t) \rightarrow f^*(t)$$
The Transfer Function (2)

- **Time invariance:**
  \[ f(t-t_0) \rightarrow f^*(t-t_0) \]

- **Linearity:**
  \[ a \cdot f_1(t) + b \cdot f_2(t) \rightarrow a \cdot f_1^*(t) + b \cdot f_2^*(t) \]

- However, almost no system is truly linear and time-invariant. But it can be useful to approximate systems as such.
The Transfer Function (3)

Building blocks of transfer functions \( g(t): x(t) \rightarrow y(t), \ G(s) \)

- **Proportional element:**
  \[
  y(t) = K_P \cdot x(t) \\
  G(s) = Y(s)/X(s) = K_P
  \]

- **Integrating element:**
  \[
  y(t) = K_I \cdot \int x(t) \, dt \\
  G(s) = K_I / s
  \]

- **Differential element:**
  \[
  y(t) = K_D \cdot dx(t)/dt \\
  G(s) = K_D \cdot s
  \]
The Transfer Function (4)

Building blocks of transfer functions \( g(t): x(t) \rightarrow y(t), \ G(s) \)

- **first order delay (PT\(_1\))**: 
  \[
  T_0 \cdot y'(t) + y(t) = K_P \cdot x(t) \\
  G(s) = \frac{K_P}{1 + s \cdot T_0}
  \]

- **second order delay (PT\(_2\))**: 
  \[
  T_{01}y''(t) + T_{02}y'(t) + y(t) = K_P \cdot x(t) \\
  G(s) = \frac{K_P}{1 + s \cdot T_{02} + s^2 \cdot T_{01}}
  \]
Testing the Transfer Function (1)

- When applying a test function the output allows to reason about the transfer function.
- Possible test functions:
  - dirac signal $\delta(t)$:
    $$\delta(t) = (\infty \text{ if } t=0, \text{ otherwise } 0), \quad \mathcal{L}\{\delta(t)\} = 1$$
  - standard step signal $\sigma(t)$:
    $$\sigma(t) = (1 \text{ if } t \geq 0, \text{ otherwise } 0), \quad \mathcal{L}\{\sigma(t)\} = 1/s$$
Testing the Transfer Function (2)

- Proportional element
  \[ y(t) = K_P \cdot x(t) \]

- Integrating element
  \[ y(t) = K_I \cdot \int x(t) \, dt \]
Testing the Transfer Function (3)

- Differential element
  \[ y(t) = K_D \cdot \frac{dx(t)}{dt} \]

- First order delay
  \[ T_0 \cdot y'(t) + y(t) = K_P \cdot x(t) \]
Testing the Transfer Function (4)

- Second order delay

(picture taken from Wikipedia)
Transfer Function of Control System

- Transfer function of parallel components:

\[ G(s) = G_1(s) + G_2(s) \]
Transfer Function of Control System (2)

- Transfer function of serial components:

\[ G(s) = G_1(s) \cdot G_2(s) \]
Transfer Function of Control System (2)

- Transfer function of feedback loop:

\[ G(s) = \frac{G_1(s) \cdot G_2(s)}{1 + G_1(s) \cdot G_2(s)} = \frac{G_0(s)}{1 + G_0(s)} \]
Stability: Nyquist Criterion

- Plotting the transfer function of the opened control system as Nyquist Plot (separate axis for real and imaginary part)
- Critical point of $F(\omega)$: \([-1, j0]\)
- Amplitude passage $\omega_{D1}$: $|F(\omega_{D1})| = 1$
- Phase passage $\omega_{D2}$: $\text{arc}(F(\omega_{D2})) = \pi$
- Stable behavior: $\omega_{D1} < \omega_{D2}$
Stability: Nyquist Criterion (2)

Critical point

\[ \omega = \infty \]

\[ \omega > \]

\[ \omega = 0 \]
Stability: Bode Plot

- $|G(j\omega)|$
- $\phi$
- $|F| > 1 \ldots$ instabil
- $|F| = 1$
- $|F| < 1 \ldots$ stabil
- $\phi > -180^\circ \ldots$ stabil
- $\phi = -180^\circ$
- $\phi < -180^\circ \ldots$ instabil
Designing a Control System in the Time Domain using the Step Response

- \( x(t) \)
- \( t \)
Designing a Control System in the Time Domain using the Step Response (2)

- $T_u$ …effective delay time
- $T_a$ …effective compensation time
- $K_s$ …gain
Designing a Control System in the Time Domain using the Step Response (3)

- Based on the values $T_u$, $T_a$ and $K_s$ the rules by [Chien, Hrones, and Reswick] can be used to find a first setting of the controller:
  - supports P, PI, and PID controller types
  - settings for 0% and 20% allowed overshooting
  - depending on the control path, fine tuning may be required.
### Rules from Chien, Hrones, and Reswick

<table>
<thead>
<tr>
<th>Controller Type</th>
<th>0% overshooting</th>
<th>20% overshooting allowed</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>$K_C^P = \frac{0.3 \cdot T_a}{K_s \cdot T_u}$</td>
<td>$K_C^P = \frac{0.7 \cdot T_a}{K_s \cdot T_u}$</td>
</tr>
<tr>
<td>PI</td>
<td>$K_C^P = \frac{0.35 \cdot T_a}{K_s \cdot T_u}$, $K_C^{\tau} = \frac{K_C^P}{1.2 \cdot T_a}$</td>
<td>$K_C^P = \frac{0.6 \cdot T_a}{K_s \cdot T_u}$, $K_C^{\tau} = \frac{K_C^P}{T_a}$</td>
</tr>
<tr>
<td>PID</td>
<td>$K_C^P = \frac{0.6 \cdot T_a}{K_s \cdot T_u}$, $K_C^{\tau} = \frac{K_C^P}{T_a}$, $K_C^D = 0.5 \cdot K_C^P \cdot T_u$</td>
<td>$K_C^P = \frac{0.95 \cdot T_a}{K_s \cdot T_u}$, $K_C^{\tau} = \frac{K_C^P}{1.35 \cdot T_a}$, $K_C^D = 0.47 \cdot K_C^P \cdot T_u$</td>
</tr>
</tbody>
</table>
Boundary conditions of Designing a Control System by Rules using the Step Response

- Requires time invariance and linearity of control path.
- The described method is for analog controllers.
- Method gives an approximate starting configuration.
- Possible optimizations either manually or using sophisticated algorithms (e.g. genetic algorithm)
Designing a Digital Controller (PID)

- **Position algorithm:**

\[ u_n = K_P e_n + K_I \sum_{i}^{n} e_i T_s + K_D \frac{e_n - e_{n-1}}{T_s} \]

- **Velocity algorithm:**

\[ \Delta u_n = K_P (e_n - e_{n-1}) + K_I e_n T_s + K_D \frac{(e_n - 2e_{n-1} - e_{n-2})}{T_s} \]

- **Sampling time** \( T_s \) **must be sufficiently small:**

\[ T_s \leq 0.1 \cdot T_a \]
\[ T_s \leq 0.25 \cdot T_u \]
Bang-Bang Control

• 2 states only (on/off) $u(t)$
• Magnitude of hysteresis $\Delta$

• Extension: three level controller
Fuzzy Logic – Fuzzy Control

- Multi-valued logic
- Instead of true or false: degree of membership to a “Fuzzy Set”
- Membership function:

\[ \mu_A(x) : X \rightarrow [0,1] \]
Example: Fuzzy Variable “temp”

- 5 fuzzy sets (very cold, cold, …)
- $\mu_{\text{cold}}(x) = 0.12$
- $\mu_{\text{moderate}}(x) = 0.33$
Operators on Fuzzy Sets

\[ \mu_{\neg A}(x) = 1 - \mu_A(x) \]

\[ \mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)] \]

\[ \mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)] \]
Process flow of fuzzy systems
Example: Steam Turbine

Input Variables: temperature and pressure

Output Variable: throttle setting
Example: Fuzzyfication

1) Temperature Input $x$:
   - $\mu_{\text{cold}}(x) = 0.6$
   - $\mu_{\text{very cold}}(x) = 0$
   - $\mu_{\text{very hot}}(x) = 0$

2) Pressure Input $y$:
   - $\mu_{\text{low}}(y) = 0.37$
   - $\mu_{\text{ok}}(y) = 0.13$
   - $\mu_{\text{weak}}(y) = 0$
   - $\mu_{\text{high}}(y) = 0$
Example: Rule Base

1) IF temperature IS cold AND pressure IS low THEN throttle IS positive

2) IF temperature IS cold AND pressure IS ok THEN throttle IS zero

3) IF temperature IS cold AND pressure IS strong THEN throttle IS negative

4) …
Example: Rule Evaluation (Max/Min)

1) IF temperature IS cold AND pressure IS low THEN throttle IS positive
   \[ \mu_{\text{cold}}(x) = 0.6 \]
   \[ \mu_{\text{low}}(y) = 0.37 \]

2) IF temperature IS cold AND pressure IS ok THEN throttle IS zero
   \[ \mu_{\text{cold}}(x) = 0.6 \]
   \[ \mu_{\text{ok}}(y) = 0.13 \]

3) IF temperature IS cold AND pressure IS strong THEN throttle IS negative

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Example: Aggregation (Max/Min)
Example: Defuzzyfication

- Generation of crisp output value by use of “center of gravity” method
Summary

• **Linear time-invariant systems** simplify control systems design.

• Time or frequency response can be used for designing control systems.

• **Requirements to control systems:**
  – Stability
  – Limited overshooting
  – Sufficient dynamics
Summary cont.

- Bang-bang or three level controllers for simple task
- **Mathematical model** may be **too difficult or expensive** to gain → **Fuzzy control** offer an alternative
- Fuzzy control requires a priori knowledge
- Fuzzy control requires 4 steps:
  - Fuzzyfication
  - Rule evaluation
  - Aggregation
  - Defuzzyfication
Dankeschön!

Any Questions?
Neural Controllers

- Learning algorithm operates *online* or *offline*
- Success is not guaranteed
Fuzzy Control vs. Neural Control

- Incorporates (expert) knowledge
- Knowledge must be gained
- Tuning is laborious and not a formal method
- Tuning possibly fails

- Learns
- Black box behaviour
- Training possibly fails

Both do not require a mathematical model of the process
Cooperative neuro-fuzzy models

- Pre- or postprocessing neural network (modifies the in- or output of the fuzzy controller)
- Neural network adjusts, or even generates, the membership functions or the rules
Hybrid neuro-fuzzy controller

Diagram showing the process of fuzzification, rule evaluation, and defuzzification.